A BISHOP-PHELPS-BOLLOBÁS THEOREM FOR THE DISC ALGEBRA

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Dedicated to the memory of Professor K. R. Parthasarathy, a wonderful person and a great mathematician

Abstract. Let $\mathbb{D}$ represent the open unit disc in $\mathbb{C}$. Denote by $A(\mathbb{D})$ the disc algebra, and $\mathscr{B}(X, A(\mathbb{D}))$ the Banach space of all bounded linear operators from a Banach space X into $A(\mathbb{D})$. We prove that, under the assumption of equicontinuity at a point in $\partial\mathbb{D}$, the Bishop-Phelps-Bollobás property holds for $\mathscr{B}(X, A(\mathbb{D}))$.

1. Introduction

Norm attainment is one of the most natural properties that bounded linear operators or functionals acting on Banach spaces can have. Given that this property is automatic for functionals or operators acting on finite-dimensional Banach spaces but not for those acting on infinite-dimensional Banach spaces, one is tempted to wonder about the behavior of norm-attaining functionals or operators acting on infinite-dimensional Banach spaces (see the survey by Aron and Lomonosov [5]). Recall that a bounded linear operator $T : X \to Y$ between Banach spaces X and Y ($T \in \mathscr{B}(X, Y)$ in short, and $T \in \mathscr{B}(X)$ if $Y = X$) is *norm attaining* if there exists a unit vector $x_0 \in S_X$ such that

$$\|T\|_{\mathscr{B}(X,Y)} = \|Tx_0\|_Y,$$

where $S_X = \{x \in X : \|x\| = 1\}$, the unit sphere of X. All the Banach spaces considered in this paper are over the field $\mathbb{C}$.

In reference to the nature of norm-attaining functionals on Banach spaces, the classical Bishop-Phelps theorem states [6]:

Theorem 1.1 (Bishop-Phelps)**.** *The set of norm-attaining functionals on a Banach space is norm dense in the dual space.*

Bollobás sharpened this amazing result, which gives a simultaneous approximation of functionals close to norm attainment at a point by norm-attaining functionals while keeping the point of norm attainment close to the given point. More specifically (cf. [11, Corollary 3.3]):

Theorem 1.2 (Bishop-Phelps-Bollobás)**.** *Let X be a Banach space, $x \in S_X$, $f \in S_{X^*}$, and let $\epsilon \in (0, 1)$. Suppose*

$$|f(x)| \geq 1 - \epsilon.$$

Then there exist $y \in S_X$ and $g \in S_{X^*}$ such that

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(1) $|g(y)| = 1$,
(2) $\|x - y\| \leq \sqrt{2\epsilon}$, *and*
(3) $\|f - g\| \leq \sqrt{2\epsilon}$.

Now we will talk about the follow-up question that comes up naturally: norm-attainment for operators on Banach spaces. Lindenstrauss [19] pioneered the study of norm-attainment or Bishop-Phelps type property for operators between Banach spaces. And finally, Acosta, Aron, García, and Maestre [1, Definition 1.1] introduced the notion of Bishop-Phelps-Bollobás property for operators on Banach spaces.

Definition 1.3. A pair of Banach spaces (X, Y) satisfies the Bishop-Phelps-Bollobás property, if for every $\epsilon > 0$, there exist $\beta(\epsilon) > 0$ and $\gamma(\epsilon) > 0$ with $\lim_{t\to 0^+} \beta(t) = 0$ such that for all $T \in S_{\mathscr{B}(X,Y)}$ and $x \in S_X$ with

$$\|Tx\| > 1 - \gamma(\epsilon),$$

there exist $y \in S_X$ and $N \in S_{\mathscr{B}(X,Y)}$ such that

(1) $\|Ny\| = 1$,
(2) $\|x - y\| < \beta(\epsilon)$, and
(3) $\|T - N\| < \epsilon$.

If $X = Y$, then we simply say that X satisfies the Bishop-Phelps-Bollobás property.

Following the case of functionals on Banach spaces, one might wonder if the Bishop-Phelps-Bollobás theorem also applies to $\mathscr{B}(X, Y)$ for Banach spaces X and Y. In fact, Lindenstrass's [19] argument shows that there are a pair of Banach spaces which do not even satisfy the Bishop-Phelps property. Since then, identifying Banach spaces that satisfies the Bishop-Phelps-Bollobás property has proven to be a challenging as well as important problem.

Let us highlight some of the noteworthy results in this direction. In [1], Acosta, Aron, García, and Maestre presented a condition on Y for which (X, Y) satisfies the Bishop-Phelps-Bollobás property for an arbitrary Banach space X. In particular, if Y is c_0 or ℓ^∞, then for an arbitrary Banach space X, the pair (X, Y) satisfies the Bishop-Phelps-Bollobás property. If $X = C(K)$ and $Y = C(M)$ for some compact Hausdorff spaces K and M, then (X, Y) satisfies the Bishop-Phelps-Bollobás property in the real case [2] (the complex case, however, is still an open question). In this regard, also see Johnson and Wolfe [17]. Furthermore, in [11, 18], it is proved that, if Y is a subalgebra of $C(K)$ or $C_b(K)$, then for an arbitrary Banach space X, the pair (X, Y) satisfies the Bishop-Phelps-Bollobás property for Asplund operators. We refer the reader to [3, 10] for more on the Bishop-Phelps-Bollobás property in different contexts.

Now we turn to the disc algebra, the central object of this paper. The disc algebra $A(\mathbb{D})$ is the commutative Banach algebra of all bounded analytic functions on the open unit disc $\mathbb{D}$ with a continuous extension on $\overline{\mathbb{D}}$. In other words

$$A(\mathbb{D}) = \mathrm{Hol}(\mathbb{D}) \cap C(\overline{\mathbb{D}}).$$

In this context, we recall that $A(\mathbb{D})$ is a closed subalgebra of $H^\infty(\mathbb{D})$, the commutative Banach algebra of bounded analytic functions on $\mathbb{D}$. The space $H^\infty(\mathbb{D})$ (as well as its closed subalgebra

$A(\mathbb{D})$) is equipped with the uniform norm

$$\|\varphi\|_\infty = \sup_{z\in\mathbb{D}} |\varphi(z)| \qquad (\varphi \in H^\infty(\mathbb{D})).$$

There are many reasons to study the disc algebra. In fact, the disc algebra is one of the most important and concrete non-selfadjoint commutative Banach algebras [8, 9, 12]. Since this space is more "tractable" (than, say, $H^\infty(\mathbb{D})$), it can be used to test a theory of interest. Finding a solution for the disc algebra appears to be a pressing issue, which is exactly what we attempt to do in this paper. However, we have not yet been able to fully settle the question for $A(\mathbb{D})$ in this circumstance. We need to assume in addition that the image of the unit sphere of the given operator satisfies the equicontinuity property at a point in $\mathbb{T}$.

Therefore, the main contribution of this paper is the fact that under the assumption of equicontinuity (see Definition 2.1 below), the Bishop-Phelps-Bollobás property of simultaneous approximations (as defined in Definition 1.3) holds for $A(\mathbb{D})$. In fact, we prove much more: under the equicontinuity assumption, $(X, A(\mathbb{D}))$ satisfies the Bishop-Phelps-Bollobás property for an arbitrary Banach space X.

A similar conclusion holds for the Asplund operators on uniform algebras [4, 11]. However, neither is $A(\mathbb{D})$ an Asplund space nor it is clear how the present equicontinuity condition is related to Fréchet differentiability and the Radon-Nikodým property, which are integral parts of the Asplund operators. On the other hand, the equicontinuity condition appears to be more practical in terms of checking the relevant conditions (cf. Section 3); yet, our result is not the best because there are norm attaining operators that fail to meet the equicontinuity characteristic (see Example 3.4).

The paper is structured as follows. Besides this section, there are two more sections in this paper. The next section addresses the main result, while the third and final section provides examples of operators that do (as well as do not) admit norms and operators on $A(\mathbb{D})$ that satisfy the main theorem's hypothesis.

2. The main result

In this section, we prove the Bishop-Phelps-Bollobás property for $A(\mathbb{D})$. We begin by recalling the definition of equicontinuity of a family of functions between metric spaces.

Definition 2.1. Let (X, d_X) and (Y, d_Y) be two metric spaces, and let $x_0 \in X$. A family of functions $\mathcal{F}$ from X to Y is said to be equicontinuous at x_0 if for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$d_Y(f(x), f(x_0)) < \epsilon,$$

for all $f \in \mathcal{F}$ and $x \in X$ such that $d_X(x, x_0) < \delta$.

Now we proceed to the main result of this paper. The starting point of the proof of the result is similar to that of [11, Lemma 2.5] and [18, Lemma 3]). Moreover, the construction of the norm attaining operator N as we do in (2.1) is again the same as the previous constructions of norm attaining operators (for instance, see [11, Equation (1.3)]). Except for them, the concept of the proof diverges considerably from that of earlier spaces as one continues.

Theorem 2.2. *Let X be a Banach space, $x_0 \in S_X$, $T \in S_{\mathcal{B}(X,A(\mathbb{D}))}$, $\epsilon \in (0,1)$, and let θ_0 be a real number. Suppose*

$$\mathcal{F} = \{Tx \mid x \in S_X\},$$

is equicontinuous at $e^{i\theta_0}$, and

$$\left|Tx_0(e^{i\theta_0})\right| > 1 - \frac{\epsilon}{3}.$$

Then there exist $y_0 \in S_X$ and $N \in S_{\mathcal{B}(X,A(\mathbb{D}))}$ such that

(1) $\|Ny_0\| = 1$,
(2) $\|x_0 - y_0\| < \sqrt{2\epsilon}$, *and*
(3) $\|T - N\| < 8\sqrt{\epsilon}$.

Proof. Since $0 < \epsilon < 1$, the Stolz region Ω_ϵ is well defined, where

$$\Omega_\epsilon := \{z \in \mathbb{C} : |z| + (1-\epsilon)|1-z| \leq 1\}.$$

By the above definition, it is evident that $0, 1 \in \Omega_\epsilon$. Moreover, there exists a homeomorphism $\psi_\epsilon : \overline{\mathbb{D}} \to \Omega_\epsilon$ (a refinement of the Riemann mapping theorem, see [20, Theorems 14.8, 14.19]) such that

(1) $\psi_\epsilon|_{\mathbb{D}}$ is a conformal mapping onto the interior of Ω_ϵ,
(2) $\psi_\epsilon(1) = 1$, and
(3) $\psi_\epsilon(0) = 0$.

By construction of Ω_ϵ, it follows that $\epsilon^2\overline{\mathbb{D}} \subseteq \Omega_\epsilon$. Moreover, since 0 is in the open set ${\psi_\epsilon}^{-1}(\epsilon^2\mathbb{D})$, there exists $\delta_1 \in (0, \epsilon)$ such that

$$\delta_1\mathbb{D} \subseteq {\psi_\epsilon}^{-1}(\epsilon^2\mathbb{D}).$$

Since $\mathcal{F} = \{Tx \mid x \in S_X\}$ is equicontinuous at $e^{i\theta_0}$, there exists $\delta_2 > 0$ such that

$$|Tx(z) - Tx(e^{i\theta_0})| < \epsilon,$$

for all $z \in \overline{\mathbb{D}}$ such that $|z - e^{i\theta_0}| < \delta_2$. Define an open set U in $\overline{\mathbb{D}}$ by

$$U = \{z \in \overline{\mathbb{D}} : |z - e^{i\theta_0}| < \delta_2\}.$$

Since $e^{i\theta_0} \in \mathbb{T}$, there exists $g_1 \in A(\mathbb{D})$ such that (see [14, page 80-81]) $g_1(e^{i\theta_0}) = 0$, and

$$\mathrm{Re} g_1(z) < 0 \qquad (z \in \overline{\mathbb{D}} \setminus \{e^{i\theta_0}\}).$$

By the continuity of g_1 on $\overline{\mathbb{D}}$, we find a number $\gamma > 0$ such that

$$\mathrm{Re} g_1(z) \leq -\gamma \qquad (z \in \overline{\mathbb{D}} \setminus U).$$

Choose $0 < \epsilon_1 < 1$ such that $\epsilon_1^\gamma < \delta_1$. Since $e^{-n} \to 0$ as $n \to \infty$, there exists $n_0 \in \mathbb{N}$ such that $e^{-n_0} < \epsilon_1$. Define

$$h(z) := e^{n_0 g_1(z)} \qquad (z \in \overline{\mathbb{D}}).$$

In view of $\mathrm{Re} g_1(z) \leq 0$, we find

$$|h(z)| = e^{n_0 \mathrm{Re} g_1(z)} \leq e^0 = 1,$$

whereas $g_1(e^{i\theta_0}) = 0$ yields $h(e^{i\theta_0}) = e^{n_0 g_1(e^{i\theta_0})} = 1$. For each $z \in \overline{\mathbb{D}} \setminus U$, we have

$$|h(z)| = e^{n_0 \mathrm{Re} g_1(z)} = e^{-n_0(-\mathrm{Re} g_1(z))} < \epsilon_1^{-\mathrm{Re} g_1(z)}.$$

From the fact that $-\text{Re}g_1(z) \geq \gamma$ for all $z \in \overline{\mathbb{D}} \setminus U$, and $0 < \epsilon_1 < 1$, we conclude that

$$|h(z)| < \epsilon_1^{\gamma}.$$

By the inequality $\epsilon_1^{\gamma} < \delta_1$, we must have

$$|h(z)| < \delta_1 \qquad (z \in \overline{\mathbb{D}} \setminus U).$$

Define η on $\overline{\mathbb{D}}$ by $\eta := \Psi_\epsilon \circ h$. Then

$$\eta(e^{i\theta_0}) = (\Psi_\epsilon \circ h)(e^{i\theta_0}) = \Psi_\epsilon(1) = 1,$$

and

$$\eta(\overline{\mathbb{D}} \setminus U) = \Psi_\epsilon(h(\overline{\mathbb{D}} \setminus U)) \subseteq \Psi_\epsilon(\delta_1 \mathbb{D}) \subseteq \epsilon^2 \mathbb{D}.$$

Moreover,

$$|\eta(z)| = |\Psi_\epsilon(h(z))| \leq |h(z)| \leq 1,$$

for all $z \in \overline{\mathbb{D}}$ yields $\|\eta\| \leq 1$. Also, for $z \in \overline{\mathbb{D}}$, we have

$$|\eta(z)| + (1-\epsilon)|1 - \eta(z)| = |\Psi_\epsilon(h(z))| + (1-\epsilon)|1 - \Psi_\epsilon(h(z))| \leq 1.$$

Define a functional $\Psi : X \longrightarrow \mathbb{C}$ by

$$\Psi x = Tx(e^{i\theta_0}) \qquad (x \in X).$$

It is easy to see that $\|\Psi\| \leq 1$. Also

$$|\Psi x_0| = |Tx_0(e^{i\theta_0})| > 1 - \frac{\epsilon}{3}.$$

Finally, define $\Psi_1 = \dfrac{\Psi}{\|\Psi\|} : X \longrightarrow \mathbb{C}$. Then $\|\Psi_1\| = 1$, and

$$|\Psi_1 x_0| = \left| \frac{\Psi}{\|\Psi\|} x_0 \right| \geq |\Psi x_0| > 1 - \frac{\epsilon}{3}.$$

Now we can apply Theorem 1.2, the Bishop-Phelps-Bollobás theorem, to Ψ_1. Consequently, there exist a vector $y_0 \in S_X$ and a linear functional $\Psi_2 : X \to \mathbb{C}$ such that $\|\Psi_2\| = 1$ and

(i) $|\Psi_2(y_0)| = 1$,
(ii) $\|y_0 - x_0\| \leq \sqrt{2\epsilon}$, and
(iii) $\|\Psi_1 - \Psi_2\| < \sqrt{2\epsilon}$.

Define a linear operator $N : X \longrightarrow A(\mathbb{D})$ by

$$(Nx)(z) := \eta(z)\Psi_2(x) + (1-\epsilon)(1 - \eta(z))Tx(z), \tag{2.1}$$

for all $x \in X$ and $z \in \mathbb{D}$. Since

$$\begin{aligned} |Nx(z)| &\leq |\eta(z)||\Psi_2(x)| + (1-\epsilon)|1 - \eta(z)||Tx(z)| \\ &\leq |\eta(z)| + (1-\epsilon)|1 - \eta(z)| \\ &\leq 1, \end{aligned}$$

for all $z \in \mathbb{D}$ and $x \in X$ with $\|x\| \le 1$, it follows that $\|N\| \le 1$. Also, note that

$$\left|Ny_0(e^{i\theta_0})\right| = \left|\eta(e^{i\theta_0})\Psi_2(y_0) + (1-\epsilon)(1-\eta(e^{i\theta_0}))Ty_0(e^{i\theta_0})\right| = |\Psi_2(y_0)| = 1.$$

Fix $x \in S_X$. Then

$$\begin{aligned}
\|(N-T)x\| &= \|\eta\Psi_2(x) + (1-\epsilon)(1-\eta)Tx - Tx\| \\
&= \|\eta\Psi_2(x) - \epsilon(1-\eta)Tx + Tx - \eta Tx - Tx\| \\
&= \|\eta(\Psi_2(x) - Tx) - \epsilon(1-\eta)Tx\| \\
&\le \|\eta(\Psi_2(x) - Tx)\| + \epsilon\|1-\eta\|\|Tx\| \\
&\le \|\eta(\Psi_2(x) - Tx)\| + 2\epsilon.
\end{aligned}$$

Next, we estimate

$$\begin{aligned}
\|\eta(\Psi_2 x - Tx)\| &= \|\eta(\Psi_2 x - \Psi_1 x + \Psi_1 x - \Psi x + \Psi x - Tx)\| \\
&\le \|\eta\|\|\Psi_2 x - \Psi_1 x\| + \|\eta\|\|\Psi_1 x - \Psi x\| + \|\eta(\Psi x - Tx)\| \\
&\le \sqrt{2\epsilon} + \left\|\frac{\Psi}{\|\Psi\|}x - \Psi x\right\| + \|\eta(Tx(e^{i\theta_0}) - Tx)\| \\
&\le \sqrt{2\epsilon} + |1 - \|\Psi\|| + \|\eta(Tx(e^{i\theta_0}) - Tx)\| \\
&\le \sqrt{2\epsilon} + \frac{\epsilon}{3} + \|\eta(Tx(e^{i\theta_0}) - Tx)\|,
\end{aligned}$$

and, finally

$$\begin{aligned}
\|\eta(Tx(e^{i\theta_0}) - Tx)\| &= \sup_{z\in\overline{\mathbb{D}}} |\eta(z)||Tx(e^{i\theta_0}) - Tx(z)| \\
&\le \sup_{z\in\overline{\mathbb{D}}\setminus U} |\eta(z)||Tx(e^{i\theta_0}) - Tx(z)| + \sup_{z\in U} |\eta(z)||Tx(e^{i\theta_0}) - Tx(z)| \\
&\le 2\epsilon^2 + \sup_{z\in U} |\eta(z)||Tx(e^{i\theta_0}) - Tx(z)| \\
&< 2\epsilon + \epsilon.
\end{aligned}$$

We conclude finally that $\|N - T\| < 8\sqrt{\epsilon}$. This completes the proof. □

In particular, we have the following Bishop-Phelps-Bollobás property for the disc algebra:

Corollary 2.3. *Let $T \in \mathcal{B}(A(\mathbb{D}))$, $f_0 \in S_{A(\mathbb{D})}$, $\|T\| = 1$, $\epsilon \in (0,1)$, and let θ_0 be a real number. Suppose*

$$\mathcal{F} = \{Tg \mid g \in S_{A(\mathbb{D})}\},$$

is equicontinuous at $e^{i\theta_0}$, and

$$\left|Tf_0(e^{i\theta_0})\right| > 1 - \frac{\epsilon}{3}.$$

Then there exist $N \in S_{\mathcal{B}(A(\mathbb{D}))}$ and $g_0 \in S_{A(\mathbb{D})}$ such that

(1) $\|Ng_0\| = 1$,
(2) $\|f_0 - g_0\| < \sqrt{2\epsilon}$, and
(3) $\|T - N\| < 8\sqrt{\epsilon}$.

We do not know how to remove the equicontinuity assumption from the preceding theorem. Furthermore, it is improbable that the equicontinuity assumption is necessary for norm attainment, but we have no evidence to support this claim either. In the following section, we will demonstrate the existence of operators that satisfy the equicontinuity condition.

3. Examples

Nontrivial examples of operators attaining or failing to attain norms have been of general interest. The goal of this final section is to provide some examples of operators in that range. We also draw a general observation concerning operator ideals of $\mathscr{B}(A(\mathbb{D}))$.

We begin with two simple (yet classical) classes of operators that attain their norms. Let $\varphi \in A(\mathbb{D})$. Define the multiplication operator M_φ and the composition operator C_φ on $A(\mathbb{D})$ by

$$M_\varphi f = \varphi f \qquad (f \in A(\mathbb{D})),$$

and

$$C_\varphi f = f \circ \varphi \qquad (f \in A(\mathbb{D})),$$

respectively. For the case of the composition operator, we must assume in addition that φ is a self-map of $\mathbb{D}$. In either case, we observe that

$$\|M_\varphi\| = \|\varphi\|_\infty = \|M_\varphi 1\|,$$

and

$$\|C_\varphi\| = 1 = \|C_\varphi 1\|.$$

Consequently, M_φ and C_φ are norm attaining operators.

Nontrivial examples of not norm-attaining operators are based on another important space, namely, the Hardy space $H^2(\mathbb{D})$. Recall that (see [13, 14])

$$H^2(\mathbb{D}) = \Big\{ f \in \mathrm{Hol}(\mathbb{D}) : \|f\|_2 := \Big(\sup_{0<r<1} \frac{1}{2\pi} \int_{\mathbb{T}} |f(rz)|^2 d\mu(z) \Big)^{\frac{1}{2}} < \infty \Big\},$$

where $d\mu$ is the normalized Lebesgue measure on $\mathbb{T}$ $(= \partial\mathbb{D})$. It is known that a function

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathrm{Hol}(\mathbb{D}),$$

is in $H^2(\mathbb{D})$ if and only if

$$\|f\|_2 = \Big(\sum_{n=0}^{\infty} |a_n|^2 \Big)^{\frac{1}{2}} < \infty.$$

Now, we are ready to provide a bounded linear operator $T : A(\mathbb{D}) \to H^2(\mathbb{D})$ which is not norm-attaining.

Example 3.1. For each $f(z) = \sum_{n=0}^{\infty} a_n z^n \in A(\mathbb{D})$, define a linear operator $T : A(\mathbb{D}) \longrightarrow H^2(\mathbb{D})$ by

$$(Tf)(z) = \sum_{n=0}^{\infty} \Big(1 - \frac{1}{n+1} \Big) a_n z^n.$$

We compute

$$\|Tf\|_2 = \left(\sum_{n=0}^{\infty}\left(1-\frac{1}{n+1}\right)^2 |a_n|^2\right)^{\frac{1}{2}} < \left(\sum_{n=0}^{\infty}|a_n|^2\right)^{\frac{1}{2}} = \|f\|_2 \leq \|f\|_\infty.$$

This implies that T is bounded as well as not norm-attaining.

It is crucial to build examples that satisfy the hypothesis of this paper's main result. In the following proposition, we do exactly that.

Proposition 3.2. *Let X be a Banach space and let $T \in \mathscr{B}(X, A(\mathbb{D}))$ be a compact operator. Suppose $\|T\| = 1$, and let*

$$\mathscr{T} = \{Tx \ : \ x \in S_X\}.$$

Then $\mathscr{T}$ is equicontinuous at each point of $\mathbb{T}$.

Proof. First, we assume that $T = F$ is a finite rank operator. There exists a linearly independent set $\{f_1, f_2, \ldots, f_n\} \subset A(\mathbb{D})$ such that

$$F(X) = \operatorname{span}\{f_1, \ldots, f_n\}.$$

For each $x \in X$, we write

$$Fx = \sum_{i=1}^{n} f_i^*(Fx) f_i,$$

where f_i^* is the i-th coordinate functional for $\{f_1, \ldots, f_n\}$. Set

$$M = \max_{1\leq i\leq n} \|f_i^*\|.$$

Fix $e^{i\theta_0} \in \mathbb{T}$ and $\epsilon > 0$. Being a finite set, $\{f_1, f_2, \ldots, f_n\}$ is equicontinuous at $e^{i\theta_0}$. Therefore, there exists $\delta > 0$ such that

$$|f_i(z) - f_i(e^{i\theta_0})| < \frac{\epsilon}{Mn},$$

for all $z \in \mathbb{D}$ such that $|z - e^{i\theta_0}| < \delta$. Now we estimate

$$\begin{aligned} |Fx(z) - Fx(e^{i\theta_0})| &= |\sum_{i=1}^{n} f_i^*(Fx) f_i(z) - \sum_{i=1}^{n} f_i^*(Fx) f_i(e^{i\theta_0})| \\ &\leq \sum_{i=1}^{n} \|f_i^*\| |f_i(z) - f_i(e^{i\theta_0})| \\ &< M\frac{\epsilon}{Mn} n \\ &= \epsilon, \end{aligned}$$

as $\|F\| = 1$. This implies that the collection $\mathscr{F} = \{Fx \ : \ x \in S_X\}$ is equicontinuous at $e^{i\theta_0}$. Now we assume that T is compact. Since $A(\mathbb{D})$ has a Schauder basis, it has the approximation property [7]. Therefore, T can be approximated by finite rank operators [21, Proposition 4.12]. Therefore, for $\epsilon > 0$, there exists a finite rank operator $F \in \mathscr{B}(X, A(\mathbb{D}))$, $\|F\| = 1$ such that

$$\|T - F\| < \frac{\epsilon}{3}.$$

Let $e^{i\theta_0} \in \mathbb{T}$. An application of the above observation on finite rank operators yields equicontinuity of $\mathscr{F} = \{Fx : x \in S_X\}$. Therefore, there exists $\delta > 0$ such that

$$|Fx(z) - Fx(e^{i\theta_0})| < \frac{\epsilon}{3},$$

for all $z \in \mathbb{D}$ and $|z - e^{i\theta_0}| < \delta$. Following the hypothesis, set $\mathscr{F} = \{Tx : x \in S_X\}$. For each $z \in \mathbb{D}$ and $x \in X$, we apply the $\epsilon/3$ argument to compute:

$$\begin{aligned}|Tx(z) - Tx(e^{i\theta_0})| &= |Tx(z) - Fx(z) + Fx(z) - Fx(e^{i\theta_0}) + Fx(e^{i\theta_0}) - Tx(e^{i\theta_0})| \\ &\leq |Tx(z) - Fx(z)| + |Fx(z) - Fx(e^{i\theta_0})| + |Fx(e^{i\theta_0}) - Tx(e^{i\theta_0})| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon.\end{aligned}$$

Therefore, we conclude that the collection $\mathscr{F}$ is equicontinuous at each point of $\mathbb{T}$. $\square$

In particular, Theorem 2.2 leads us to conclude that compact operators from X to $A(\mathbb{D})$ can be approximated by norm-attaining operators. We remark that this conclusion also follows from [11, Theorem 3.6].

The following particular case may be of some interest. Suppose X is a non-reflexive Banach space. By the James theorem [15, 16], there exists a linear functional $x_0^* \in X^*$ which fails to attain the norm. Fix a non-zero function h in $A(\mathbb{D})$. Define $T \in \mathscr{B}(X, A(\mathbb{D}))$ by

$$Tx = x_0^*(x)h \qquad (x \in X).$$

It is easy to see that T is not norm attaining. Moreover, by the above proposition, it follows that

$$\mathcal{F} = \{x_0^*(x)h : x \in S_X\}$$

is equicontinuous at every point of $\mathbb{T}$, and hence, by Theorem 2.2, T can be approximated by norm-attaining operators.

Finally, an example of a norm attaining operator on $A(\mathbb{D})$:

Example 3.3. Let $x^* \in A(\mathbb{D})^*$, $\|x^*\| = 1$, be a norm attaining functional. There exists $f_0 \in S_{A(\mathbb{D})}$ such that $|x^*(f_0)| = 1$. Fix a nonzero $h \in A(\mathbb{D})$. Define $T \in \mathscr{B}(A(\mathbb{D}))$ by

$$Tf = x^*(f)h,$$

for all $f \in A(\mathbb{D})$. A routine computation ensures that $\|T\| = \|x^*\|\|h\|$. Then

$$\|Tf_0\| = \|x^*(f_0)h\| = |x^*(f_0)|\|h\| = \|x^*\|\|h\| = \|T\|,$$

implying that T is norm attaining.

The above example, in particular, applies to evaluation functionals. For a fixed $z_0 \in \mathbb{D}$, define the evaluation functional $ev_{z_0} \in A(\mathbb{D})^*$ by

$$ev_{z_0}f = f(z_0) \qquad (f \in A(\mathbb{D})).$$

Clearly

$$\|ev_{z_0}\| = 1 = |ev_{z_0}(1)|,$$

that is, ev_{z_0} is norm attaining. Moreover, $\|T1\| = \|T\|$, where T is defined as in the above example.

The following example illustrates that there are operators on $A(\mathbb{D})$ which are norm attaining but not equicontinuous.

Example 3.4. Consider the identity operator $I : A(\mathbb{D}) \to A(\mathbb{D})$. This is clearly a norm-attaining operator. However, $\{f \in A(\mathbb{D}) : \|f\| = 1\}$ is not equicontinuous at any point on the unit circle. To see this, fix θ and let $\epsilon = \frac{1}{2}$. It is easy to see that $\|z^n\|_{A(\mathbb{D})} = 1$ for all $n \in \mathbb{N}$. Now for each $\delta > 0$, one can find $z \in \mathbb{D}$ with $|z - e^{i\theta}| < \delta$ such that

$$|z^n - (e^{i\theta})^n| > \frac{1}{2},$$

for large n. This proves that the identity operator I on $A(\mathbb{D})$ is norm attaining but not equicontinuous at any point on the unit circle.

Now we turn to the operator ideals of $\mathscr{B}(A(\mathbb{D}))$. Given a Banach space X, a subspace $\mathscr{I} \subseteq \mathscr{B}(X)$ is an *operator ideal* if $\mathscr{I}$ contains all operators of finite rank and

$$T_1 \circ T \circ T_2 \in \mathscr{I},$$

for all $T \in \mathscr{I}$ and $T_1, T_2 \in \mathscr{B}(X)$.

Corollary 3.5. *Let $\mathscr{I} \subseteq \mathscr{B}(A(\mathbb{D}))$ be an operator ideal, $T \in \mathscr{I}$, $\|T\| = 1$, $f_0 \in S_{A(\mathbb{D})}$, $\epsilon \in (0,1)$, and let θ_0 be a real number. Suppose $\mathcal{F} = \{Tg : g \in S_{A(\mathbb{D})}\}$ is equicontinuous at $e^{i\theta_0}$, and*

$$\left|Tf_0(e^{i\theta_0})\right| > 1 - \frac{\epsilon}{3}.$$

Then there exist $N \in \mathscr{I}$ and $g_0 \in S_{A(\mathbb{D})}$ such that

(1) $\|Ng_0\| = \|N\| = 1$,
(2) $\|f_0 - g_0\| < \sqrt{2\epsilon}$, and
(3) $\|T - N\| < 6\epsilon + \sqrt{2\epsilon}$.

Proof. By applying Corollary 2.3 to T, it is enough to show that $N \in \mathscr{I}$, where N is defined as in (2.1):

$$(Ng)(z) := \eta(z)\Psi_2(g) + (1-\epsilon)(1-\eta(z))Tg(z),$$

for all $g \in A(\mathbb{D})$ and $z \in \mathbb{D}$. We write $N = N_1 + N_2$, where

$$(N_1 g)(z) = \eta(z)\Psi_2(g),$$

and

$$(N_2 g)(z) = (1-\epsilon)(1-\eta(z))Tg(z),$$

for all $g \in A(\mathbb{D})$ and $z \in \mathbb{D}$. Since $\mathscr{I}$ is an ideal, it is enough to prove that $N_1, N_2 \in \mathscr{I}$. We see that $N_1 \in \mathscr{I}$ as N_1 is a rank one operator. Consequently, $T \in \mathscr{I}$ gives us $N_2 \in \mathscr{I}$, and completes the proof. □

The same result holds for weakly compact operators.

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